

# Computation and Incentives in Combinatorial Public Projects

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## ABSTRACT

The Combinatorial Public Projects Problem (CPPP) is an abstraction of resource allocation problems in which agents have preferences over alternatives, and an outcome that is to be collectively shared by the agents is chosen so as to maximize the social welfare. We explore CPPP from both computational and mechanism design perspectives. We examine CPPP in the hierarchy of complement-free (subadditive) valuation classes and present positive and negative results for both unrestricted and truthful algorithms.

## Categories and Subject Descriptors

F.2 [Analysis of Algorithms and Problem Complexity]: ; J.4 [Social and Behavioral Science]: [Economics]

## General Terms

Algorithms, Theory, Economics

## Keywords

Algorithmic mechanism design

## 1. INTRODUCTION

The *Combinatorial Public Project Problem* (CPPP), introduced in [13], is a resource allocation problem in which a set of resources is chosen to be collectively shared by a group of agents, so as to maximize the agents' social welfare. An instance of CPPP consists of  $n$  agents  $\{1, \dots, n\}$ ,  $m$  resources

$\{1, \dots, m\}$ , a valuation function  $v_i : 2^{[m]} \rightarrow \mathcal{R}_{\geq 0}$  for each agent  $i$  mapping each set of resources to his value for that set, and an integer  $k \in [m]$ . The goal is to choose a set  $S$  of  $k$  resources for which the total social welfare  $\sum_i v_i(S)$  is maximized. CPPP captures committee elections and network design, among other problems (see [13, 16]).

Exploring the boundaries of tractability in CPPP is natural when approaching questions from a computational perspective. Furthermore, this boundary plays a crucial role in algorithmic mechanism design [12]: for social welfare maximization problems, such as CPPP, computational tractability implies a computationally-efficient truthful implementation via the celebrated VCG mechanism. Intractability, on the other hand, can make truthful implementations with good approximation ratios impossible to obtain. This was recently demonstrated in [13], where it was shown for the first time, in the context of CPPP with submodular agents, that even if constant approximation ratios are achievable if one only cares about computational efficiency *or* truthfulness, combining *both* desiderata can lead to non-constant lower bounds.

Our aim here is to enable an understanding of CPPP both purely computationally and with truthfulness as an added constraint. For the case of unrestricted agent valuations, CPPP is known to be NP-hard to approximate well [16]. Hence, seeking interesting special cases of CPPP for which reasonable approximation ratios are attainable is natural. We consider the case where agents' valuation functions are *complement-free*, *i.e.*, cases in which agents' valuations are *subadditive* set functions. The class of complement-free, or subadditive, valuations encapsulates a rich hierarchy of valuation functions [9, 11] (see Fig. 1.1), that has been the focal point of the study of approximability in combinatorial auctions (see, *e.g.*, [2, 5, 6, 9, 17]).

In our study of the *computational feasibility* of CPPP, we search for two thresholds:

1. the point in the complement-free hierarchy of agents' valuations at which CPPP ceases to be tractable, and hence, for which computationally-efficient truthful implementation is no longer achievable via VCG mechanisms;
2. the point at which CPPP ceases to be approximable within a constant factor (*i.e.*, not in APX), and so

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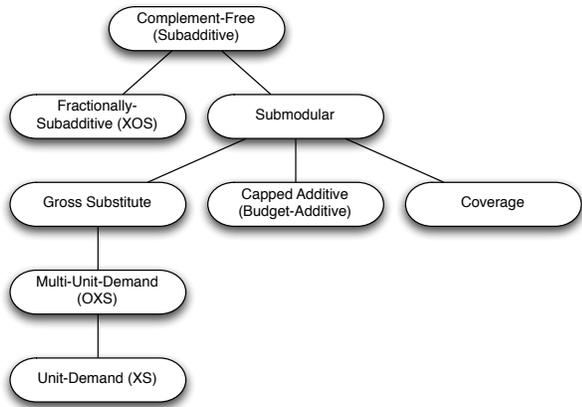


Figure 1.1: The Complement-Free Hierarchy

CPPP cannot be well approximated even from a purely computational perspective (disregarding incentives).

In our study of *incentives* in CPPP, we focus both on general truthful mechanisms and on the broad class of “VCG-based”, or “maximal-in-range”, mechanisms, that have received much attention in algorithmic mechanism design recently (see [2, 3, 8, 13]).

CPPP is closely related to combinatorial auctions, which have been studied extensively across the computer science, microeconomics and operations research disciplines over past decades. To date, despite much research, many big questions regarding the interplay between computation and truthfulness in combinatorial auctions and other related games remain wide open. We believe that CPPP is a more suitable arena for exploring the fundamental connections between computation and truthfulness. Indeed, the advances in [13] and [7], as well as our results, demonstrate the amenability of CPPP to an algorithmic mechanism design examination.

## 1.1 Our Results

We now briefly and informally survey our main results and their implications:

**Is CPPP tractable?** For CPPP with  $n$  agents, we show that even for the lowest (most restricted) class of valuations in the complement-free hierarchy, finding an *optimal* outcome is NP-hard. Specifically, CPPP is hard even for “unit-demand” valuations, in which every agent is only interested in getting a single resource. Moreover, going up just one step higher in the hierarchy, CPPP becomes hard even for a constant number of agents.

On the positive side, we present an optimal (and truthful via VCG payments) algorithm for an interesting special case of CPPP.

**Is CPPP approximable?** Our main inapproximability result is for *fractionally-subadditive* valuations. We show that, unlike the case of CPPP with submodular valuations [13], for fractionally-subadditive valuations, no constant approximation ratio is achievable (unless P is contained in quasi-NP). Our result is nearly tight [16], answers an open question from [16], and is the first non-constant *computational* complexity lower bound for this class (and for subadditive valuations in general).

We present many other positive and negative approximability results: We show that the  $1 - \frac{1}{e}$  approximation ratio for CPPP with submodular valuations [13] is tight even for unit-demand valuations. By contrast, we present improved ratios for other well-studied subclasses of submodular valuations.

**Truthful mechanisms for CPPP.** We present both truthful mechanisms and hardness results for truthful computation. In particular, we present an inapproximability result for truthful mechanisms for CPPP that both strengthens and greatly simplifies the result in [13]. Surprisingly, our result holds even for the case of a single agent, thus raising an intriguing question in algorithmic mechanism design. We also present a truthful constant-approximation mechanisms for interesting special cases of CPPP.

Finally, we present several inapproximability results for the class of “VCG-based”, or “maximal-in-range”, truthful mechanisms. In particular, we show that no constant approximation ratio is achievable for such mechanisms even with unit demand valuations. Interestingly, we show a truthful constant-factor approximation for CPPP with unit-demand agents, thus establishing a gap between VCG-based and general truthful mechanisms.

Our results are summarized in the tables below (Fig. 1.2 and 1.3, which also suggest interesting directions for future research).

## 1.2 Organization of the Paper

Each of the sections 2-6 focuses on exactly one class in the complement-free hierarchy. In Sec. 2 we present our results for unit-demand valuations. Sec. 3, 4, 5 and 6, deal with multi-unit-demand valuations, capped additive valuations, coverage valuations, and fractionally-subadditive valuations, respectively. We conclude and discuss our results in Sec. 7.

## 2. UNIT-DEMAND: TRUTHFULNESS AND COMPLEXITY

**Unit-demand valuations.** The simple class of *unit demand* valuations, in which every agent is only interested in getting a single resource, constitutes the lowest level of the complement-free hierarchy (see [9, 11], where unit-demand valuations are termed “XS”).

**DEFINITION 2.1 (UNIT-DEMAND VALUATIONS).** A valuation  $v$  is called *unit demand* if  $v(S) = \max_{i \in S} v(\{i\})$ , for every  $S \subseteq [m]$ . Such a valuation is represented by a list of the  $m$  values  $v(\{j\})$ ,  $j \in [m]$ .

Our results in this section shall be proven for an even more restrictive class of valuations: unit demand valuation such that  $v(\{i\}) \in \{0, 1\}$  for each resource  $i$ , and each agent has a value of 1 for at most 2 resources. We shall refer to this subclass of unit-demand valuations as “2- $\{0, 1\}$ -unit-demand”.

**THEOREM 2.1.** CPPP with  $n$  2- $\{0, 1\}$ -unit-demand valuations is NP-hard to solve optimally. Furthermore, no algorithm for CPPP with  $n$  unit-demand valuations has an approximation ratio of  $1 - \frac{1}{e} + \epsilon$  unless  $P=NP$  (for any constant  $\epsilon > 0$ ).

**PROOF.** We will show an approximation preserving reduction from MAX- $t$ -COVER. The problem of MAX- $t$ -COVER

valuation class	no. of agents	appx. ratio $r$
unit-demand	constant	$r = 1$
	$n$	$r = 1 - \frac{1}{e}$ [New]
multi-unit-demand	1, 2	$r = 1$ [New]
	3	$2/3$ [New] $\leq r < 1$ [New]
	$\geq 4$	$1 - \frac{1}{e}$ [10] $\leq r < 1$ [New]
	$\geq 10$	$1 - \frac{1}{e}$ [10] $\leq r < 1 - \epsilon$ (FPTAS)[New]
	$n$	$r = 1 - \frac{1}{e}$ [New]
capped additive	1	$r = 1$
	constant $\geq 2$	$r = 1 - \epsilon$ (FPTAS) [New]
	$n$	$r = 1 - \frac{1}{e}$ [New]
fractionally-subadditive	constant	$r = 1$
	$n$	$\max\{\frac{1}{n}, \frac{1}{\sqrt{m}}\}$ [16] $\leq r \leq 2^{-\frac{\log^{1-\gamma} n}{6}}$ [New]

Figure 1.2: Computational Results

valuation class	no. of agents	Truthful appx. ratio $r$	VCG-based appx. ratio $r$
2- $\{0,1\}$ unit-demand	$n$	$1/2 \leq r < 1$ [New]	$r = \frac{1}{\sqrt{m}}$ [New]
unit-demand	$n$	?	
multi-unit-demand	3	$2/3 \leq r < 1$ [New]	
	$n$	?	
capped-additive	$\geq 2$	?	
coverage	1	$r = \frac{1}{\sqrt{m}}$ [New]	
fractionally-subadditive	$n$	?	

Figure 1.3: Truthful Mechanism Results. Question marks indicate that the only bounds known are a  $\frac{1}{\sqrt{m}}$  lower bound based on the VCG-based mechanism shown in [16] and a purely computational upper-bound.

takes as input a collection of subsets  $\mathcal{F}$  of a set  $A$  and an integer  $t$ . The goal is to find  $t$  sets in  $\mathcal{F}$  which have a union of maximum cardinality. It was shown in [4] that MAX- $t$ -COVER cannot be approximated to within  $1 - 1/e + \epsilon$  for any constant  $\epsilon > 0$  unless  $P = NP$ .

Consider a MAX- $t$ -COVER instance over set  $A$  with  $\mathcal{F} = \{S_1, \dots, S_\ell\}$  and number of sets to be chosen  $t$ . We create a CPPP instance with resource set  $\mathcal{F}$  and  $|A|$  agents, one corresponding to each element of  $A$ . The agent corresponding to element  $a$  values each resource  $S_i \in \mathcal{F}$  as

$$v_a(S_i) = \begin{cases} 1, & a \in S_i \\ 0, & \text{otherwise} \end{cases}.$$

So the value for agent  $a$  is 1 if  $a$  is covered by the chosen set and 0 otherwise. Thus, the social welfare is number of covered resources, or the cardinality of the union of the chosen sets. By setting the number of resources allowed to be chosen to  $k = t$ , we see that if we can approximate the social welfare to within any factor  $\alpha$ , we get an  $\alpha$ -approximation of MAX- $t$ -COVER as well. So by [4], an approximation of  $1 - 1/e + \epsilon$  is not achievable.  $\square$

Observe that the above hardness of approximation result is tight (a simple greedy algorithm obtains an approximation ratio of exactly  $1 - \frac{1}{e}$ ). Note that the above proof required  $|\mathcal{F}|$  agents, each with very simple 0/1 valuation functions. Observe that if there is only a constant number  $c$  of agents, one need only consider  $\binom{m}{\min(c,k)}$   $\in \text{poly}(m)$  sets of resources in order to find one which maximizes the social welfare, and hence CPPP with a constant number of unit-demand agents can be solved in polynomial time.

**VCG-based mechanisms.** We next consider the class of VCG-based, or *maximal-in-range* (MIR), mechanisms. For a thorough explanation about MIR/VCG-based mechanisms, see [13]. Informally, MIR mechanisms output, for each possible input, the *optimal* outcome within a *fixed* set of outcomes. That is, a MIR mechanism  $M$  has a *fixed* set  $R$  of possible outcomes (subsets of resources of size  $k$ ) and, for each  $n$ -tuple of agents' valuations  $(v_1, \dots, v_n)$ , chooses a subset  $r \in R$  that maximizes the social welfare  $\sum_i v_i$  over  $R$ . The collection  $R$  is called  $M$ 's *range*.

[16] shows a computationally-efficient MIR mechanism for CPPP with subadditive valuations that has an approximation ratio of  $\frac{1}{\sqrt{m}}$ . This approximation ratio is tight for MIR mechanisms even when restricted to 2- $\{0,1\}$ -unit-demand valuations.

**THEOREM 2.2.** *No computationally-efficient MIR mechanism can approximate CPPP with  $n$  2- $\{0,1\}$ -unit-demand valuations within  $m^{-(\frac{1}{2}-\epsilon)}$  (for any constant  $\epsilon > 0$ ) unless  $NP \subseteq P/\text{poly}$ .*

**PROOF.** Our proof is based on the proof technique in [13], where the use of VC dimension to set bounds on the approximability of MIR mechanisms is introduced. The reader is referred to [13] for a comprehensive explanation. We begin by noting that in [13] it was shown that any algorithm for CPPP which achieves an approximation ratio of at least  $m^{1/2-\epsilon}$  has a range of size  $\Omega(e^{m^\epsilon})$ . This proof required that for any  $V \subseteq [m]$ , it is possible to create a set of agents such that the social welfare is  $v(S) = |V \cap S|$ . This is easy to do with  $n$  2- $\{0,1\}$ -unit-demand agents, resulting in the following useful lemma:

LEMMA 2.3. *If a maximal-in-range mechanism for CPPP with  $n$  2- $\{0,1\}$ -unit-demand agents achieves an approximation ratio of at least  $m^{1/2-\epsilon}$ , it must have a range of size  $\Omega(e^{m^\epsilon})$ .*

From this, we can use the Sauer-Shelah lemma to see that the range has a VC dimension at least  $m^\alpha$  for some constant  $\alpha > 0$ . This large range allows us to perform reductions similar to the ones we use in our NP-hardness proofs to show inapproximability. We begin with the modified unit-demand reduction.

As shown above, any maximal-in-range mechanism which approximates better than  $m^{1/2-\epsilon}$  must have a range with VC-dimension at least  $m^\alpha$ . Re-order the resources such that the  $m^\alpha$  corresponding to this VC-dimension are the set  $[m^\alpha]$ . We show a reduction from vertex cover with  $m^\alpha/2$  edges. Let  $k' < k$  be the target size of the vertex cover. The first  $|V|$  resources correspond to the vertices.

The first  $2|E| = m^\alpha$  agents correspond 2 to each edge, and have valuation 1 if the corresponding edge is covered by one of the vertices corresponding to a resource chosen from  $[m^\alpha]$ , we have  $m - m^\alpha$  agents, one corresponding to each resource outside of  $[m^\alpha]$  where the agent corresponding to resource  $i$  has valuation

$$v_i(S) = \begin{cases} 1, & i \in S \\ 0, & \text{otherwise} \end{cases} .$$

If a single edge is unsatisfied, more social welfare can be obtained by adding a resource from  $[m^\alpha]$ , where some resource adds at least 2 to the social welfare than by adding a resource from  $[m] - [m^\alpha]$ , which only contributes 1. So if the minimum vertex cover has size  $k^*$ , the maximum social welfare is  $2|E| + (k - k')$ . Furthermore,  $\mathcal{M}$  will find this maximum, as its range includes every subset of  $[m^\alpha]$ , padded out with arbitrary elements from  $[m] - [m^\alpha]$  to reach size  $k$ . Thus,  $\mathcal{M}$  can be used to find the size of the minimum vertex cover and therefore cannot run in polynomial time unless  $NP \subset P/poly$ .  $\square$

**General truthful mechanisms.** Theorem 2.2 shows that no constant-approximation MIR mechanisms exist even for CPPP with 2- $\{0,1\}$ -unit-demand valuations. In contrast, a simple non-adaptive greedy algorithm achieves an approximation ratio of  $\frac{1}{2}$  for 2- $\{0,1\}$ -unit-demand valuations that is truthful without payments, thus establishing a large gap between what is achievable via MIR and general truthful mechanisms.

THEOREM 2.4. *There exists a computationally efficient, truthful mechanism for CPPP with  $n$  agents with 2- $\{0,1\}$ -unit-demand valuations that achieves an approximation ratio of  $\frac{1}{2}$ .*

PROOF. **The mechanism :**

1. for each resource  $j$  let  $s_j = |\{i : v_i(\{j\}) = 1\}|$ .
2. sort the  $m$  resources in decreasing order by the value of  $s_j$ , breaking ties in favor of resources with lower indices.
3. output the set  $S$  consisting of the  $k$  first resources in the above ordering.

This is clearly an efficient algorithm, as sort only requires  $O(n \log n)$  time, and in this case bucket sort can be used to achieve a linear time mechanism.

The mechanism as described essentially allows agents to vote for 2 resources, then chooses the  $k$  with the most votes. An agent only benefits from adding votes to the 2 resources that he actually desires, as adding other resources to the top  $k$  does not improve his social welfare. As the two resources are desired equally, there is no advantage to voting for one of the resources the bidder desires and not the other. So there is never an incentive for an agent to not declare his valuation truthfully.

We will now see that this has an approximation ratio of  $1/2$ . Every agent has a value of either 0 or 1 for the chosen set. If an agent has a value of 1, we call that agent satisfied. For each resource  $j$ , let  $s_j$  be the number of agents satisfied by  $j$ . For any set  $T$ ,  $\sum_{j \in T} s_j$  is an upper bound on the social welfare of  $T$ . Clearly,  $S$  maximizes  $\sum_{j \in S} s_j$  for sets of size  $k$ , so  $\sum_{j \in S} s_j$  is an upper bound on the maximum social welfare. Furthermore, each agent is satisfied by at most 2 resources in  $S$ , so the social welfare of  $S$  is at least  $\sum_{j \in S} s_j / 2 = 1/2 \sum_{j \in S} s_j$ , which is at least  $1/2$  the maximum social welfare.  $\square$

### 3. MULTI-UNIT-DEMAND: AN OPTIMAL MECHANISM FOR 2 AGENTS

**Multi-unit-demand valuations.** Multi-unit-demand valuations (termed ‘‘OX’’ in [9, 11]) are a generalization of unit-demand valuations.

DEFINITION 3.1 (MULTI-UNIT-DEMAND). *A valuation  $v$  is a multi-unit-demand valuation if there exist unit demand valuations  $\{v^1, \dots, v^w\}$  such that, for every  $S \subseteq [m]$ ,*

$$v(S) = \max_{P = \{P^1, \dots, P^w\}} \sum_{r \in [w]} v^r(P^r)$$

where the above maximum is taken over all  $w$ -partitions  $P = \{P^1, \dots, P^w\}$  of  $S$ . Such a valuation agent is represented by a list of the  $w$  unit demand valuations.

**How hard is CPPP with multi-unit-demand valuations?** Unit-demand valuations are a subclass of multi-unit-demand valuations, and so our negative results in Sec. 2 for CPPP with  $n$  agents extend to multi-unit-demand valuations. What about a constant number of agents? One can easily show that CPPP with a single multi-unit-demand valuation is optimally solvable in a computationally-efficient manner using maximum matching on a bipartite graph. Below, we shall prove that CPPP with 2 multi-unit-demand valuations is tractable. We now present the following hardness result for CPPP with 3 multi-unit-demand valuations:

THEOREM 3.1. *CPPP with 3 multi-unit-demand valuations is NP-hard to solve optimally.*

PROOF. We reduce from 3-Dimensional Matching (3DM). Given a 3DM instance  $M \subseteq [q] \times [q] \times [q]$ , the goal is to determine whether there exists a set  $M' \subset M$  of size  $q$  such that no two members of  $M'$  share a coordinate. Our reduction is as follows. The set of resources is  $M$ . The number of resources to be chosen is  $k = q$ . If there is a set of size  $q$  such that no two members share a coordinate, then there should be  $q$  different values for each coordinate in the set.

We will simply create an agent for each coordinate that has a valuation equal to the number of distinct values seen in that coordinate, so that the social welfare is maximized if no two resources coincide on any coordinate.

The  $i$ th agent values set  $S$  by the number of different values for the  $i$ th coordinate in set  $S$ . This valuation is multi-unit-demand because it can be built out of the  $q$  unit-demand valuations that value 1 to any set containing a resource with a  $j$  in the  $i$ th coordinate, for any  $1 \leq j \leq q$ . The maximum possible value is  $q$  (1 from each unit-demand valuation), and this is only achievable if every possible value of the  $i$ th coordinate appears in  $S$ .

The maximum social welfare of this auction is  $3q$  iff the 3DM instance is positive. Clearly, any set  $M'$  of size  $q$  will have social welfare  $3q$  iff none of the resources in the set share a coordinate, as the maximum value of  $q$  is achieved by each agent none of the resources share the coordinate corresponding to that agent.  $\square$

While the above theorem leaves open the possibility of a PTAS for any constant number of agents, we can rule out this possibility by presenting a hardness of approximation result for 10 (or more) agents.

**THEOREM 3.2.** *There exists a constant  $\epsilon > 0$  such that it is NP-hard to approximate the social welfare in CPPP with 10 multi-unit-demand valuations within a ratio of  $1 - \epsilon$ .*

**PROOF.** Consider an instance of MAX-3SAT-5 consisting of  $\ell$  clauses,  $c_1, \dots, c_\ell$ . Because each clause has 3 variables and each variable is contained in 5 clauses, there are  $3\ell/5$  variables  $v_1, \dots, v_{3\ell/5}$ . We will start by reducing to an instance with  $n$  unit-demand agents, then demonstrate that these agents can be compressed into 10 multi-unit-demand agents without changing the social welfare function.

There are  $6\ell/5$  items, 2 corresponding to each variable. For each variable  $v_i$ , we will have two items labeled  $i$  and  $\bar{i}$ . Choosing  $i$  corresponds to setting  $v_i$  to true, while choosing  $\bar{i}$  corresponds to setting  $v_i$  to false. We allow  $k = 3\ell/5$  items to be chosen, so one value can be chosen for each variable.

There are two classes of agents. The first class has  $\ell$  agents, one corresponding to each clause. The agent corresponding to clause  $c_i$  has value 1 for each item  $j$  such that  $v_j$  is in  $c_i$  and 1 for each item  $\bar{j}$  such that  $\neg v_j$  is in  $c_i$ . Thus, these agents have value 1 if their clause is satisfied and 0 otherwise.

The second class of agents has  $5 \cdot (3\ell/5) = 3\ell$  agents, 5 for each variable. The 5 agents for each variable are identical, and the agents corresponding to  $v_i$  have value 1 for items  $i$  and  $\bar{i}$  and 0 for all other items. If there is some item  $i$  for which both  $i$  and  $\bar{i}$  are chosen, then by the pigeonhole principle, there is some  $j$  for which neither  $j$  nor  $\bar{j}$  is chosen. This leads to a loss of 5 to the social welfare from these agents compared to replacing one of  $i, \bar{i}$  with one of  $j, \bar{j}$ , while having both  $i$  and  $\bar{i}$  adds at most 5 to the social welfare of the clause agents compared to keeping just one of these. So these agents allow us to modify any choice of items such that only one of  $i, \bar{i}$  is chosen for each  $i$  without reducing the social welfare. Thus, we will assume without loss of generality that any assignment of items corresponds to an assignment to the variables of the MAX-3SAT-5 instance. Note that this means that the social welfare from the second class of agents is exactly  $3\ell$  in any assignment we consider.

It was shown in [4] that there exists a positive constant  $\delta$  such that it is NP-hard to distinguish between the case that

all clauses are satisfiable in a MAX-3SAT-5 instance and that only a  $1 - \delta$  fraction are. Let  $\epsilon$  be a positive constant less than  $\delta/4$ . If the instance of MAX-3SAT that we are reducing from is one in which all  $\ell$  clauses are satisfiable, then in the produced CPPP instance, we can choose the corresponding items, arriving at  $\ell$  social welfare from the clause agents and  $3\ell$  social welfare from the rest for a total social welfare of  $4\ell$ . So if it is possible to approximate the social welfare to a factor of  $1 - \epsilon$ , we will find a social welfare of at least  $(1 - \epsilon)4\ell > 4\ell - \delta\ell$ .

As mentioned above,  $3\ell$  of the social welfare comes from the second class of agents described, so more than  $(1 - \delta)\ell$  social welfare comes from the clause agents. This shows that there is an assignment satisfying more than a  $1 - \delta$  fraction of the clauses, allowing us to distinguish between the case that all clauses are satisfiable and the case that at most a  $1 - \delta$  fraction are satisfied. As this is NP-hard, we have shown that approximating the social welfare for these unit-demand agents to within  $1 - \epsilon$  is NP-hard as well.

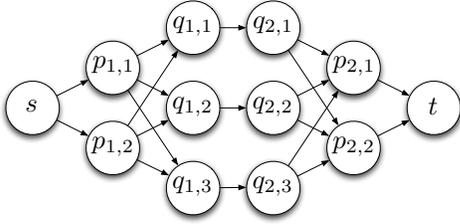
Finally, we show how these  $4\ell$  unit-demand agents can be compressed into 10 multi-unit-demand agents. We do so by combining groups of unit-demand agents that don't value any of the same items into a single multi-unit-demand agent. Then the multi-unit-demand value to that agent of any set is the sum of the values of each of the individual unit-demand agents, as we can partition the items according to which unit-demand valuation has value for each item. Note that we can assume without loss of generality that there is no  $i$  for which the clause agents never value  $i$  (or similarly,  $\bar{i}$ ), as we could simply remove all agents valuing  $i$  or  $\bar{i}$ , then perform a  $1 - \epsilon$  approximation with  $k' = k - 2$  and add the  $i$  back knowing whether to choose  $i$  or  $\bar{i}$  and resulting in an improved approximation. Thus, each item  $i$  or  $\bar{i}$  is only valued by at most 4 clause agents.

Start with 10 multi-unit-demand agents with value 0 for all sets of items. We add the unit-demand valuations to them greedily, beginning with the valuations of the clause agents. For each unit-demand clause agent, simply add its valuation to any multi-unit-demand agent which does not yet value any of the 3 items it values. Since each item  $i$  or  $\bar{i}$  is valued by at most 3 other clause agents and there are three items valued by any clause agent, there are at most  $3 \cdot 3 = 9$  multi-unit-demand clause agents with values for these 3 items. Thus, one of the 10 multi-unit-demand agents can accommodate the value of this unit-demand agent.

Now, we add the second class of agents, each of which values items  $i$  and  $\bar{i}$  at 1 for some  $i$ . There are 5 such agents for each  $i$ . Similarly, for each  $\bar{i}$ , there are 5 unit-demand agents which value either  $i$  or  $\bar{i}$ . So the 5 agents from the second class can be added to the valuations of the 5 multi-unit-demand agents that do not yet value items  $i$  or  $\bar{i}$  from the clause valuations.

Thus, we can compress these valuations into 10 multi-unit-demand agents while preserving the social welfare of every assignment. As the social welfare is preserved, it remains NP-hard to approximate the social welfare to within a factor of  $1 - \epsilon$ .  $\square$

**Optimal mechanism for CPPP with 2 agents.** We now show that CPPP with 2 multi-unit-demand valuations can be optimally solved in a computationally efficient manner via minimum cost flow. Thus, the use of VCG payments implies the existence of an optimal truthful mechanism.



**Figure 3.1: example of minimum-cost flow network construction for  $w = 2$  and  $m = 3$ .**

**THEOREM 3.3.** *There exists an optimal computationally-efficient and truthful mechanism for CPPP with 2 multi-unit-demand valuations.*

Our mechanism uses minimum cost flow, for which integral solutions can be found in polynomial time [14]. Minimum cost flow is similar to network flow, except that each edge has a cost and the goal is to find a flow of value  $f$  with minimum total edge cost. We now formally describe the mechanism:

**Input:** an instance of CPPP with two agents 1, 2, where each agent  $i$  has a multi-unit-demand valuation  $v_i$  such that each  $v_i$  is composed of  $w_i$  unit-demand valuations  $v_i^1, \dots, v_i^{w_i}$  (see Def. 3.1).

**The mechanism:**

1. **Step I: add “dummy” unit-demand valuations** (that equal 0 for all subsets of resources) if necessary to ensure that  $w_1 = w_2 = w \geq k$ .
2. **Step II: create a minimum-cost flow network** (see Fig. 3.1). In addition to the source and target nodes  $s$  and  $t$ , the network contains node  $p_{i,r}$  corresponding to agent  $i$ 's  $r$ th unit-demand valuation  $v_i^r$ , and two nodes  $q_{1,j}$  and  $q_{2,j}$  for each resource  $j \in [m]$ . The edge set contains an edge from  $s$  to each node  $p_{1,r}$ , and an edge from each node  $p_{2,r}$  to  $t$ . In addition, for each  $j \in [m]$ , create an edge from  $q_{1,j}$  to  $q_{2,j}$ . Set the cost of each of these edges to be 1.  
Let  $v_{max}$  be a positive real value that is strictly higher than both agents' values for each single resource (say,  $v_{max} = \max_{i \in [2], j \in [m]} v_i(\{j\}) + 1$ ). Create, for each  $j \in [m], r \in [w]$ , an edge from  $p_{1,r}$  to  $q_{1,j}$  of cost  $v_{max} - v_1^r(\{j\})$  and an edge from  $q_{2,j}$  to  $p_{2,r}$  of cost  $v_{max} - v_2^r(\{j\})$ . (Observe that all costs are positive integers.)  
Set the capacities of all edges to be 1.
3. **Step III: compute a minimum-cost flow  $f$**  with flow value  $k$  and integer flow along each edge (*i.e.*, the flow along each edge is in  $\{0, 1\}$ ).
4. **Step IV: set  $S$  to be the subset of  $[m]$  such that  $j \in S$  iff the flow in  $f$  along the edge from  $q_{1,j}$  to  $q_{2,j}$  is positive.** Observe that the  $k$  units of flow in  $f$  emanating from  $s$  must traverse exactly  $k$  edges of the form  $(q_{1,j}, q_{2,j})$ , and hence  $|S| = k$ .
5. **Step V: output the set of resources  $S$  and the VCG payments.**

Clearly, the mechanism is computationally efficient (recall that the computation of minimum-cost flow with integer values is tractable [14]). We are left with showing that the mechanism outputs the social-welfare maximizing subset of resources, after which truthfulness follows from the use of VCG payments.

**LEMMA 3.4.** *The above mechanism which solves CPPP with 2 multi-unit-demand valuations outputs a subset of resources of size  $k$  that maximizes the social welfare.*

**PROOF.** Observe that the  $k$  units of flow in  $f$  emanating from  $s$ , and the  $k$  units of flow going into  $t$  traverse edges that have a total cost of  $2k$ , and that the  $k$  units of flow along the edges from  $q_{1,j}$  nodes to  $q_{2,j}$  nodes traverse edges that have a total cost of  $k$ . Hence, the total cost of these edges is  $3k$  regardless of how the flow  $f$  is achieved.

Consider  $j \in S$  (computed in Step IV of the mechanism). Observe that because there is 1 unit of flow traversing the edge  $(q_{1,j}, q_{2,j})$ , there must be exactly one incoming edge leading to node  $q_{1,j}$ , and exactly one outgoing edge leaving node  $q_{1,j}$ , on which the flow in  $f$  is 1. Consider a specific edge  $(q_{1,j}, q_{2,j})$  and let  $(p_{1,r}, q_{1,j})$  and  $(q_{2,j}, p_{2,r'})$  be the edges through which the flow in  $f$  equals 1. Observe that the total cost of these two edges is  $2v_{max} - v_1^r(\{j\}) - v_2^{r'}(\{j\})$ . We define  $c : S \rightarrow \mathbb{Z}^+$  to be the function that maps each  $j \in S$  to the total cost of the incoming and outgoing edges to  $q_{1,j}$  and  $q_{2,j}$  (not including the edge between them).

Now, for some pair of partitions of  $S$   $P_1 = (P_1^1, \dots, P_1^w)$  and  $P_2 = (P_2^1, \dots, P_2^w)$ ,

$$\begin{aligned} \sum_{j \in S} c(j) &= 2kv_{max} - \sum_{r=1}^w v_1^r(P_1^r) - \sum_{r=1}^w v_2^r(P_2^r) \\ &\geq 2kv_{max} - \max_{P=(P^1, \dots, P^w)} \sum_{r=1}^w v_1^r(P^r) \\ &\quad - \max_{P=(P^1, \dots, P^w)} \sum_{r=1}^w v_2^r(P^r) \\ &= 2kv_{max} - v_1(S) - v_2(S), \end{aligned}$$

where the maxima in the above equations are taken over  $w$ -partitions of  $S$ .

Therefore, the total cost of flow  $f$  (including the edges leaving  $s$ , the edges entering  $t$  and the edges leading from the  $q_{1,j}$ 's to the  $q_{2,j}$ 's) is at least  $2kv_{max} + 3k$  minus the social welfare of the set  $S$ . This lower bound is tight, as choosing the set maximizing the social welfare and the incoming and outgoing flows that correspond to the unit-demand valuations that maximize each  $v_i$  guarantees a total cost of exactly  $2kv_{max} + 3k$  minus the maximum social welfare. Hence, the computation of the  $k$ -flow of minimum cost determines the value of the social-welfare maximizing outcome, and the set  $S$  produced achieves this maximum.  $\square$

Note that we can use this mechanism in a randomized fashion with 3 or more agents by selecting 2 of the agents uniformly at random, then running the mechanism on them. This gives a decent approximation ratio for 3 agents.

**COROLLARY 3.5.** *There is a randomized universally truthful mechanism for CPPP with 3 multi-unit-demand agent that achieves a 2/3 approximation of the social welfare in expectation.*

**VCG-based mechanisms.** Theorem 3.3 shows that there is a computationally efficient VCG mechanism for CPPP with 2 multi-unit-demand valuations. In contrast, we prove the following hardness of approximation result for CPPP with 3 or more multi-unit-demand valuations.

**THEOREM 3.6.** *No computationally efficient MIR mechanism can approximate CPPP with 3 multi-unit-demand valuations within  $m^{-(\frac{1}{2}-\epsilon)}$  (for any constant  $\epsilon > 0$ ) unless  $NP \subset P/poly$ .*

**PROOF.** The proof here is essentially the same as that of Theorem 2.2, in that the proof of NP-hardness can be modified to make use of the smaller range by setting each of the resources outside of  $[m^\alpha]$  to add  $1/2$  to the social welfare. In this case, we can fold this value in to the multi-unit-demand valuation of one of the agents without affecting the social welfare gain from the resources in  $[m^\alpha]$ . Running  $\mathcal{M}$  in this case yields a social welfare of  $3q + (k - q)/2$  iff there is a set of  $q$  resources from  $[m^\alpha]$  with social welfare  $3q$ , corresponding to a 3-dimensional matching. So  $\mathcal{M}$  cannot run in polynomial time unless  $NP \subset P/poly$ .  $\square$

#### 4. FPTAS FOR CPPP WITH CAPPED ADDITIVE VALUATIONS

**Capped additive valuations.** Intuitively, a capped additive valuation is a valuation function that is additive (the value for each bundle of resources is the additive sum of the per-resource values) but cannot exceed some threshold.

**DEFINITION 4.1 (ADDITIVE VALUATIONS).** *We call the valuation function  $v$  additive (linear) if  $v(S) = \sum_{j \in S} v(\{j\})$  for every  $S \subseteq [m]$ .*

**DEFINITION 4.2 (CAPPED ADDITIVE VALUATIONS).**  *$v$  is a capped additive valuation if there exists an additive valuation  $a$ , and a real value  $B > 0$ , such that, for each  $S \subseteq [m]$ ,  $v(S) = \min\{a(S), B\}$ .*

**NP-hardness and an FPTAS.** 2- $\{0,1\}$ -unit-demand valuations are a subclass of capped additive valuations (where  $B = 1$ ), and so our negative results in Sec. 2 for CPPP with  $n$  agents extend to capped additive valuations. What about a constant number of agents? Observe that finding the optimal outcome for a single agent is trivially in P (simply take the  $k$  most valued resources). It turns out, by reduction from Subset Sum, that even with 2 agents, this is no longer the case.

**THEOREM 4.1.** *CPPP with 2 capped additive valuations is NP-hard.*

**PROOF.** We reduce from Subset Sum, where we are given a set of positive integers  $v_1, \dots, v_\ell$  and a target  $t$ , and the goal is to find a subset of  $v_1, \dots, v_\ell$  that sums to  $t$ . Given an instance of Subset Sum, we construct an instance to our problem with  $m = 2\ell$  resources,  $k = \ell$ , and 2 agents with valuations  $v_1(S) = \min\{\sum_{j \in S} 2\bar{v}_{1j}, 2t\}$  and  $v_2(S) = \min\{\sum_{j \in S} \bar{v}_{2j}, B\}$ , where  $B = k \cdot \max_j v_j$  and  $\bar{v}_{1j}, \bar{v}_{2j}$  are:

$$\bar{v}_{1j} = \begin{cases} 2v_j, & j \leq \ell \\ 0, & \text{otherwise} \end{cases}$$

$$\bar{v}_{2j} = \begin{cases} B/k - v_i, & j \leq \ell \\ B/k, & \text{otherwise} \end{cases}$$

Observe that if there exists a subset  $S$  s.t.  $\sum_{i \in S} a_i = t$ , by choosing the set of resources  $S' = S \cup \{\ell + 1 \dots 2\ell - |S|\}$  we have  $v_1(S') + v_2(S') = B + t$ .

Conversely, consider a subset of resources in our problem of size  $m$  with social welfare of at least  $B + t$ . Consider the set of resources with index at most  $\ell$ . If the corresponding resources summed to more than  $t$ , then agent 2 would have total value less than  $B - t$ , while agent 1 would have value of only  $2t$ , for a total value of less than  $B + t$ . If the corresponding resources summed to less than  $t$ , then the social welfare would be  $B$ , plus the sum of the corresponding set, which is less than  $B + t$ . So the subset must have a sum of exactly  $t$ .  $\square$

However, using dynamic programming we obtain an FPTAS for any constant number of agents.

**THEOREM 4.2.** *There exists an FPTAS for CPPP with a constant number of capped additive valuations.*

**PROOF.** We will use a dynamic programming procedure. For  $b = \max_{i \in n} b_i$ , we divide the interval  $[0, b]$  into  $\frac{n \cdot m}{\epsilon}$  segments, each of length  $\frac{\epsilon b}{n \cdot m}$ , and denote  $p(x) = \lfloor x \cdot mn / \epsilon b \rfloor$ . We will maintain an  $n$  dimensional table with  $(\frac{n \cdot m}{\epsilon})^n$  entries, denoted  $A$ , where in each entry  $a_{i,j,\dots,k}$  we will store a subset  $S$  for which  $p(v_n(S)) = k, p(v_2(S)) = j, \dots, p(v_1(S)) = i$ , if such a subset exists. For every subset  $S$ ,  $A(S)$  shall denote its corresponding entry in the table.

Assume some arbitrary ordering  $\{1 \dots m\}$  over the set of resources, and consider the following procedure. We initialize the table with the empty set in all entries. At stage  $j$ , for each subset  $S \in A$ , s.t.  $|S| < k$ , let  $T = A(S \cup \{j\})$ . If  $|S \cup \{j\}| \leq |T|$  or  $T = \emptyset$ , we set  $A(S \cup \{j\}) = S \cup \{j\}$ . After the  $m$ th stage we iterate over all entries in the table, and choose the subset with highest social welfare. The procedure runs in  $O(m \cdot (\frac{mn}{\epsilon})^n)$  steps, which is polynomial in  $m$  and  $1/\epsilon$  as required.

Let  $O$  denote the optimal solution,  $O_j = \{i \leq j | i \in O\}$ . By induction on the stage of the algorithm, we can show that at stage  $\ell$  there is a subset  $S_\ell$  s.t.  $S_\ell \in A(O_\ell)$ ,  $|S_\ell| \leq |O_\ell|$  and for every agent  $i$  we have that  $v_i(O_\ell) - v_i(S_\ell) \leq \ell \cdot \frac{\epsilon b}{m \cdot n}$ . For  $\ell = 1$  the claim is trivial. For a  $\ell \leq m$ , if  $\ell \notin O_\ell$ , the claim trivially holds from the inductive hypothesis. Otherwise, there is a subset  $S_{\ell-1}$ , s.t.  $|v_i(O_\ell) - v_i(S_{\ell-1} \cup \{\ell\})| = |v_i(O_{\ell-1} \cup \{\ell\}) - v_i(S_{\ell-1} \cup \{\ell\})| \leq (\ell - 1) \cdot \frac{\epsilon b}{m \cdot n}$  for every  $i$ , and  $|S_{\ell-1}| \leq |O_{\ell-1}|$ . If another subset  $S' \neq S \cup \{\ell\}$  is stored in  $A(O_\ell)$  then  $|v_i(S \cup \{\ell\}) - v_i(S')| \leq \frac{\epsilon b}{m \cdot n}$ ,  $|S'| \leq |S \cup \{\ell\}|$ , and the claim holds.  $\square$

**VCG-based mechanisms.** In a similar fashion to the above lower bounds, we can show that there is no hope in VCG-based mechanisms for this class of valuations either.

**THEOREM 4.3.** *No computationally-efficient MIR mechanism can approximate CPPP with 2 capped additive valuations within  $m^{-(\frac{1}{2}-\epsilon)}$  (for any constant  $\epsilon > 0$ ) unless  $NP \subset P/poly$ .*

**PROOF.** Since the reduction for budget additive agents did not rely on minimizing the number of resources required to match the social welfare of the set of all resources, we cannot rely on the same trick here as in the above two theorems. Instead, we rely on the structure of the reduction. The number of resources which are valued by agent 1 at 0 and agent 2 at  $B/k$  doesn't affect the proof (as long as it's

larger than  $\ell$  and at least  $k$ ), so we just add  $m - m^\alpha$  more of these. The particular value of  $k$  also doesn't matter, as long as it's at least  $\ell$ , so losing control of how  $k$  relates to  $m^\alpha$  isn't an issue. Thus, using the same reduction after this modification, we see that  $\mathcal{M}$  can be used to solve subset sum instances of size  $m^\alpha$ , and is therefore does not run in polynomial time unless  $NP \subseteq P/poly$ .  $\square$

## 5. TRUTHFULNESS IS HARD EVEN WITH A SINGLE AGENT

**Coverage valuations.** Intuitively, in a coverage valuation each resource represents a set of elements in some universe  $U$ , and the value of each set of resources  $S \subseteq [m]$  equals the cardinality of the subset of  $U$  that is covered by its resources (that is, by the subsets of  $U$  represented by the resources in  $S$ ).

**DEFINITION 5.1 (COVERAGE VALUATIONS).** *We call the valuation function  $v$  a coverage valuation if there exists a universe of elements  $U$ , subsets  $S_1, \dots, S_m \subseteq U$ , and a real number  $\alpha > 0$  such that  $v(S) = \alpha |\bigcup_{j \in S} S_j|$ , for every  $S \subseteq [m]$ . Such a valuation is represented by a list of the  $m$  sets  $S_1, \dots, S_m$ .*

**The hardness of being truthful with a single agent.** [13] shows that no computationally-efficient and truthful mechanism for CPPP with 2 submodular valuations can obtain an approximation ratio better than  $\frac{1}{\sqrt{m}}$  (while a constant non-truthful approximation exists). [16] proves the tightness of this result in [13] by presenting a truthful, computationally efficient mechanism with an approximation ratio of  $\frac{1}{\sqrt{m}}$ .

One might suspect that the hardness of truthful computation in [13] stems from the conflict of interests between the two agents. When there is only one agent, the interests of the mechanism designer and the single agent are trivially aligned; both strive to better the agent's outcome (that is also the total social welfare). We now give the first evidence that, surprisingly, algorithmic mechanism design can be non-trivial even in single-player environments. We believe that this result raises intriguing questions in algorithmic mechanism design regarding the "right" solution concept/framework for analyzing incentives.

**Inapproximability result.** We strengthen the result in [13] (which we prove for  $n = 1$  and for coverage valuations). Our proof greatly simplifies the long and complicated proof in [13].

**THEOREM 5.1.** *No computationally-efficient and truthful mechanism for CPPP with one coverage valuation obtains an approximation ratio within  $m^{-(\frac{1}{2}-\epsilon)}$  (for any constant  $\epsilon > 0$ ) unless  $NP \subseteq P/poly$ .*

**PROOF.** We begin by presenting a simple characterization of truthful mechanisms for CPPP with a single agent, that can easily be generalized to hold for all 1-player mechanism design environments. Our characterization shows that every truthful mechanism is an *affine maximizer* (see [15, 13]).

**LEMMA 5.2.** *If  $M$  is a truthful mechanism for CPPP with a single agent 1, then there exists a collection  $O$  of subsets of resources of size  $k$ , and a real number  $w_o \in \mathbb{R}$  for each  $o \in O$  such that, for each valuation function  $v_1$  of agent 1, the outcome  $o(v_1) \in O$  that  $M$  outputs for  $v_1$  is in  $\arg\max_{o \in O} (v_1(o) - w_o)$ .*

**PROOF.** Let  $O$  be the collection of outcomes (subsets of  $[m]$  of size  $k$ ) that agent 1 can achieve (*i.e.*, outcomes that the mechanism outputs for some valuation of 1). In truthful mechanisms, the payment of a player is independent of his own valuation function, and can only depend on the outcome and on the valuations of the other players. Because we are dealing with a single-player environment, we can associate each outcome  $o \in O$  with the payment that  $M$  outputs for that outcome  $w_o$ . Now,  $M$ 's truthfulness implies that, if  $M$  outputs the outcome  $o \in O$  for the valuation  $v_1$ , then it must hold  $v_1(o) - w_o \geq v_1(o') - w_{o'}$  for each  $o' \in O$  (otherwise, 1 is better off lying and announcing the valuation for which  $M$  outputs  $o'$ ). The lemma follows.  $\square$

We now prove an inapproximability result for truthful mechanisms. Let  $M$  be a truthful mechanism. We now know that there exists a collection  $O$  of subsets of resources of size  $k$ , and per-outcome "weights" (the  $w_o$ 's), such that  $M$  exactly optimizes 1's value over  $O$ , given the outcome weights. Let  $\alpha = \max_{o \in O} 2|w_o| + 1$ . Observe that for every  $S, T \in O$  such that  $v_1(S) \neq v_1(T)$  it holds that  $|v_1(S) - v_1(T)| > |w_S - w_T|$ . So we can safely ignore the outcome weight, as maximizing  $v_1(S) - w_S$  also maximizes  $v_1(S)$ . Hence, from now on, we need only consider MIR mechanisms (that for each possible  $v_1$  output an outcome  $o(v_1) \in O$  that is in  $\arg\max_{o \in O} v_1(o)$ ). The following lemma concludes the proof of the theorem.

**LEMMA 5.3.** *No MIR mechanism for CPPP with with one coverage valuation achieves an approximation ratio of at least  $m^{-(\frac{1}{2}-\epsilon)}$  unless  $NP \subseteq P/poly$ .*

**PROOF.** Let  $M$  be a MIR mechanism that obtains an approximation ratio better than  $m^{-(\frac{1}{2}-\epsilon)}$ . [13] shows that the VC dimension of  $M$ 's range (that is,  $O$ ) must be at least  $m^\alpha$  for some constant  $\alpha > 0$  (using essentially the arguments used in the proof of Theorem 2.2).

We now show a reduction from the NP-hard  $t$ -COVER [4] with  $m^\alpha$  sets. In  $t$ -COVER, the input is  $m^\alpha$  subsets of a universe  $E, T_1, \dots, T_{m^\alpha}$ , and an integer  $t$ , and the objective is to determine whether there are  $t$  sets that cover  $E$ . We now construct the valuation function of agent 1. We create a universe  $U$  that consists of two disjoint copies of  $E, E_1$  and  $E_2$ , plus a set of  $m - t$  additional elements,  $E_3$ . To define the coverage valuation  $v_1$  we need to define the sets  $S_1, \dots, S_m \subseteq U$  (see Def. 5.1). Re-order the resources such that the  $m^\alpha$  resources corresponding to this VC-dimension are the set  $[m^\alpha]$ . For each  $j \in [m^\alpha]$  let the set  $S_j$  be the subset of  $U$  that covers all elements in  $E_1$  and  $E_2$  that are covered by  $T_j$ . For each  $j \in \{m^\alpha + 1, \dots, m\}$  let  $S_j$  be a set that covers a single unique element in  $E_3$ .

Observe that if the minimal number of sets needed to cover  $E$  in  $t$ -COVER is  $r$ , then any optimal outcome in our CPPP instance is one that contains  $r$  resources corresponding to  $r$  covering sets in  $t$ -COVER, and  $k - r$  additional resources from  $E_3$  (chosen arbitrarily). The output of  $M$  thus determines the value of  $r$ . If  $r \leq t$  then there exist  $t$  sets in MAX- $t$ -COVER that cover  $E$ , otherwise no such  $t$  sets exist. Observe that the reduction is polynomial, yet is not uniform (because of the non-constructiveness of the Sauer-Shelah Lemma), and hence our result is dependent on the computational assumption that  $NP$  is not contained in  $P/poly$ .  $\square$

## 6. INAPPROXIMABILITY OF FRACTIONALLY-SUBADDITIVE VALUATIONS

**Fractionally-subadditive valuations.** Intuitively, a valuation is fractionally-subadditive (termed “XOS” in [9, 11]) if it is the maximum of a collection of additive (linear) valuations (valuations where the value of each set is simply the additive sum of the per-resource values).

**DEFINITION 6.1** (FRACTIONALLY SUBADDITIVE). *A valuation function  $v$  is fractionally subadditive if there exist additive valuations  $\{a^1, \dots, a^t\}$  s.t.  $v(S) = \max_{j \in t} a^j(S)$ . Such a valuation is represented by a list of the  $t$  valuations  $a^j, j \in [t]$ .*

**Remark: constant number of agents.** Although multi-unit demand agents are a special case of fractionally subadditive agents in general, this leads to an exponential blowup in description size with our choice of representation. So while 3 multi-unit demand agents creates an NP-hard problem, a constant number of fractionally subadditive agents allow a polynomial time algorithm.

**THEOREM 6.1.** *CPPP with a constant number of agents with fractionally-subadditive valuations can be solved in polynomial time.*

**PROOF.** Each fractionally-subadditive valuation takes the maximum over linearly many additive valuations. If one of these additive valuations is chosen for each agent, the resulting auction can be trivially solved in polynomial time. This solution gives a lower bound on the maximum social welfare. If the additive valuations chosen happen to be the ones that exhibit the maximum in an optimal allocation, the solution found will also be optimal. Thus, by enumerating over all possible choices, an optimal allocation can be found. If there are  $c$  agents with at most  $\ell$  additive valuations each, there are  $O(\ell^c) \subseteq \text{poly}(\ell)$  choices to enumerate over. Thus, the solution to the auction can be found in polynomial time.  $\square$

**Inapproximability result.** We now present a reduction from LABEL-COVER<sub>max</sub> to CPPP with  $n$  fractionally-subadditive valuations that preserves an approximation gap. First, we define LABEL-COVER<sub>max</sub> and discuss the complexity of its approximation. A LABEL-COVER<sub>max</sub> instance consists of a regular bipartite graph  $G = (V_1, V_2, E)$ , a set of  $n$  labels  $N = \{1, \dots, n\}$  and for each edge  $e \in E$  a partial function  $\Pi_e : N \rightarrow N$ . We say that the edge  $e = \{x, y\}$  for  $x \in V_1, y \in V_2$  is satisfied if  $x$  is labeled with  $l_1$  and  $y$  with  $l_2$  such that  $\Pi_e(l_1) = l_2$ . The goal of LABEL-COVER<sub>max</sub> is to find an assignment of labels to the nodes in  $V_1$  and  $V_2$  such that each node has exactly one label and as many edges as possible are satisfied. It was shown in [1] that LABEL-COVER<sub>max</sub> is quasi-NP-hard to approximate.

**THEOREM 6.2** ([1]). *For any sufficiently small constant  $\gamma > 0$ , it is quasi-NP-hard to distinguish between the following two cases in LABEL-COVER<sub>max</sub>: (1) YES case: all edges are covered, and (2) NO case: at most a  $2^{-\log^{1-\gamma} n}$  fraction of the edges are covered, where  $n$  is the size of the LABEL-COVER<sub>max</sub> instance.*

We make use of Theorem 6.2 to show a similar hardness result for CPPP with fractionally-subadditive valuations.

**THEOREM 6.3.** *It is quasi-NP-hard to obtain an approximation ratio of  $2^{\frac{\log^{1-\gamma} b}{6}}$  for CPPP with fractionally-subadditive valuations where  $b$  is the size of the CPPP instance.*

**PROOF.** We prove this using a gap-preserving reduction from LABEL-COVER<sub>max</sub>: We are given as input an instance of LABEL-COVER<sub>max</sub> consisting of a graph  $G = (V_1, V_2, E)$ , a set of labels  $N$  and a set of partial functions  $\Pi_e$  for each  $e \in E$ . We create a CPPP instance with  $|V_1|$  agents, one corresponding to each node in  $V_1$ . The resource set is  $V_2 \times N$ . We now define the fractionally-subadditive valuation  $v_i$  of each agent  $i$ . For every label  $l \in N$ , we define the additive valuation function  $a_{i,l}$ .

$$a_{i,l}(\{(j, l')\}) = \begin{cases} 1, & \{i, j\} \in E \text{ and } \Pi_{\{i,j\}}(l) = l' \\ 0, & \text{otherwise} \end{cases}$$

So  $a_{i,l}(S)$  represents how many edges incident with  $i$  are covered if we choose label  $l$  for vertex  $i \in V_1$  and the best label from the set  $\{l' : (j, l') \in S\}$  for vertex  $j \in V_2$ .

The fractionally-subadditive valuation of agent  $i$  is defined by

$$v_i(S) = \max_{l \in N} \{a_{i,l}(S)\}.$$

So agent  $i$  gets the value for the best possible choice of a single label for vertex  $i$  given the label choices for  $V_2$  implied by  $S$ . We set the size of the set of resources to be chosen in our CPPP instance to be  $|V_2|$ .

If the LABEL-COVER<sub>max</sub> instance is a YES case, we can find a set of resources with social welfare  $|E|$ . Simply take any labeling that covers every edge and for every  $j \in V_2$ , choose the resource  $(j, l')$ , where  $j$  is labeled by  $l'$  in the labeling. Call this set  $S$ . Clearly,  $v_i(S)$  equals the degree of node  $i$ , as if we choose  $l$  such that  $i$  is labeled by  $l$ ,  $\Pi_{\{i,j\}}(l) = l'$  for each  $(j, l') \in S$ . So the social welfare given these resources is  $|E|$ .

We now show that if the LABEL-COVER<sub>max</sub> instance is a NO case, then the maximum social welfare is bounded by  $2^{-\frac{\log^{1-\gamma} n}{6}} |E|$  for sufficiently large  $n$ . Note that if  $n'$  is the size of the LABEL-COVER<sub>max</sub> instance, our construction guarantees  $n \leq (n')^2$ . So our bound is at least

$$2^{-\frac{\log^{1-\gamma} [(n')^2]}{6}} |E| \geq 4 \cdot 2^{-\frac{\log^{1-\gamma} n'}{3}} |E|$$

for sufficiently large  $n'$ . In order to simplify our expressions in the rest of the proof, let  $\alpha = 2^{-\frac{\log^{1-\gamma} n}{6}}$ . Using the above bound, we see

$$\alpha \geq 4 \cdot 2^{-\frac{\log^{1-\gamma} n'}{3}}. \quad (1)$$

Suppose by way of contradiction that we reduced from a NO case, but the maximum social welfare is at least  $\alpha|E|$ .

Let  $S$  be a set of resources in the CPPP instance with a social welfare of at least  $\alpha|E|$ . Recall also that each agent  $i$ 's fractionally-subadditive valuation  $v_i$  is defined as the pointwise maximum over a set of additive valuations. Let  $a_{i,l}$  be the additive valuation in the set of valuations making up  $v_i$  for which  $a_{i,l}(S)$  is maximized (and so  $v_i(S) = a_{i,l}(S)$ ). If we fix a choice of  $j$ ,  $a_{i,l}$  assigns a value of 1 to at most one of the resources  $(j, l')$  for  $l' \in N$ . Moreover,  $a_{i,l}$  can only assign value to a resource  $(j, l')$  if  $\{i, j\} \in E$ . We say that an edge between vertex  $i \in V_1$  and vertex  $j \in V_2$  is satisfied by the set  $S$  if  $(j, \Pi_{\{i,j\}}(l)) \in S$ . Observe that the total social welfare value of  $S$  equals the number of edges satisfied by  $S$ .

Let  $d$  be the number of incoming edges of a vertex in  $V_2$ . Since  $G$  is a regular bipartite graph,  $d = \frac{|E|}{|V_2|}$ . Let  $V_2'$  denote all vertices  $v \in V_2$  in which the number of edges incident on  $v$  satisfied by  $S$  is at least  $\frac{\alpha}{2}d$ . A counting argument shows that  $|V_2'| \geq \frac{\alpha}{2}|V_2|$ . If  $|V_2'|$  were less than  $\frac{\alpha}{2}|V_2|$ , the number of satisfied edges incident upon vertices in  $V_2'$  is at most  $|V_2'|d < \frac{\alpha}{2}|E|$ , and the number of satisfied edges incident upon vertices outside of  $V_2$  would be less than  $|V_2|\frac{\alpha}{2}d = \frac{\alpha}{2}|E|$ . So summing these, we would see that the number of satisfied edges is less than  $\alpha|E|$ , a contradiction. So  $|V_2'| \geq \frac{\alpha}{2}|V_2|$ .

If  $S$  contains  $\ell$  resources of the form  $(j, l)$  for a fixed  $j$  and  $\ell$  different values  $l \in N$ , we say that  $j$  is labeled  $\ell$  times by  $S$ . Since there are  $|S| = |V_2|$  resources, at most  $\frac{\alpha}{4}|V_2|$  of the nodes  $j \in V_2$  are labeled more than  $\frac{4}{\alpha}$  times by  $S$ . So letting  $V_2''$  be the subset of  $V_2'$  which is labeled at most  $\frac{4}{\alpha}$  times by  $S$ ,  $|V_2''| \geq \frac{\alpha}{4}|V_2|$ .

Since  $S$  labels each  $j \in V_2''$  at most  $\frac{4}{\alpha}$  times, and  $S$  satisfies at least  $\frac{\alpha}{2}d$  edges incident upon each vertex in  $V_2''$ , we can find a single  $s_j \in S$  that satisfies at least  $\frac{\alpha/2}{4/\alpha}d = \frac{\alpha^2}{8}d$  of the edges incident upon  $j$ . So if we label each  $j \in V_2''$  according to  $s_j$  and label each  $i \in V_1$  by the  $l$  such that  $v_i(S) = a_{i,l}(S)$ , we have a labeling that satisfies at least  $|V_2''|\frac{\alpha^2}{8}d = \frac{\alpha^3}{32}|E|$  edges, regardless of how the vertices in  $V_2 - V_2'$  are labeled. This contradicts that we had a NO case, as we can see by (1) that  $\frac{\alpha^3}{32}|E| > 2^{-\log^{1-\gamma} n'}|E|$ .

Thus, we see that the maximum social welfare of our CPPP is at least  $|E|$  if we reduced from a YES case and at most  $\alpha|E|$  if we reduced from the NO case. Therefore, it is quasi-NP-hard to achieve an approximation ratio of  $2^{-\frac{\log^{1-\gamma} n}{6}}$ , proving the theorem.  $\square$

## 7. DISCUSSION AND OPEN QUESTIONS

In our exploration of CPPP we have presented positive and negative results for truthful and unrestricted computation. Our results highlight interesting phenomena in algorithmic mechanism design, and improve our understanding of the tractability-intractability boundary for this natural computational and economic environment.

The focus in algorithmic mechanism design is on the tension between computation and truthfulness. Our results for CPPP identify extremely simple combinatorial environments where the two desiderata clash, and that are therefore a natural arena for the investigation of the complex interplay between computational efficiency and incentive compatibility.

We leave many important questions wide open. We still lack a good understanding of the power of computationally-efficient and truthful mechanisms for CPPP (see Fig. 1.3), and leave bridging the approximability gaps between the upper and lower bounds in Fig. 1.2 as an open question.

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